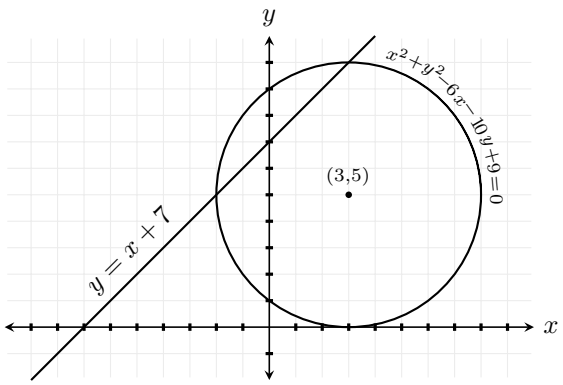
6.13 Finding The Point(s) Of Intersection Of A Circle And A Line

EXAMPLE:
Find the points of intersection of the circle $x^2+y^2-6x-10y+9=0$ and the line $x-y=-7$.



Solve both the equation of the line and the equation of the circle simultaneously:
For the line: express one variable in terms of another, **then substitute** into equation of the circle.

$$x - y = -7 \qquad \Rightarrow y = x + 7$$

$$\begin{aligned}
\text{Substituting: } & x^2 + (x+7)^2 - 6x - 10(x+7) + 9 = 0 && \ldots \text{ we end up with a quadratic,} \\
\Rightarrow\ & x^2 + (x+7)(x+7) - 6x - 10x - 70 + 9 = 0 && \text{expressed in terms of } x\text{, alone.} \\
\Rightarrow\ & x^2 + x^2 + 7x + 7x + 49 - 6x - 10x - 70 + 9 = 0 && \\
\Rightarrow\ & 2x^2 - 2x - 12 = 0 && \\
\Rightarrow\ & x^2 - x - 6 = 0 && \ldots \text{ dividing both sides by 2} \\
\Rightarrow\ & (x+2)(x-3) = 0 && \\
\Rightarrow\ & x = -2 \quad \text{and} \quad x = 3 &&
\end{aligned}$$

Substitute your obtained x coordinates into equation of **the line**, not the circle.[vii]

$$\begin{aligned}
y &= x + 7 \\
y &= (-2) + 7 \\
&= 5 \\
y &= (3) + 7 \\
&= 10
\end{aligned}$$

~~$x^2 + y^2 - 6x - 10y + 9 = 0$~~ **~~Wrong:~~**

$$\begin{aligned}
(-2)^2 + y^2 - 6(-2) - 10y + 9 &= 0 & (3)^2 + y^2 - 6(3) - 10y + 9 &= 0 \\
4 + y^2 + 12 - 10y + 9 &= 0 & 9 + y^2 - 18 - 10y + 9 &= 0 \\
y^2 - 10y + 25 &= 0 & y^2 - 10y &= 0 \\
(y-5)(y-5) &= 0 & (y-10)(y-0) &= 0 \\
y = 5 \quad \text{and} \quad y &= 5 & y = 10 \quad \text{and} \quad y &= 0
\end{aligned}$$

$\therefore$ The points of intersection are $(-2, 5)$ and $(3, 10)$. (Compare with graph, above.)

[vii] It will work for both the line and the circle equations, because both points are on the line and the circle, concurrently. When you plug a single coordinate into the equation of a circle, it can give you two answers, however. That's because you are effectively plugging in the equation of a horizontal or vertical line e.g. "$y=0$", "$x=-2$" or "$x=3$", and there can be two places of intersection.